JOURNAL OF FUNCTIONAL ANALYSIS 19, 373-389 (1975)

Annihilators of Rational Modules

ANTHONY G. O'FARRELL*

University of California, Los Angeles, California 90024

Communicated by the Editors

Received May 27, 1974

We apply the Cauchy transform to derive results which relate approximation problems in different Lipshitz norms, and in the uniform norm, to one another.

1

Let X be a compact subset of the complex plane $\mathbb{C}$, and let $\beta > 0$. This paper concerns approximation in $\operatorname{Lip}(\beta, X)$ norm by elements of the module $\mathscr{R}(X)\,\bar{\mathscr{P}}_m$, which consists of all functions of the form

$$r_0(z) + r_1(z)\bar{z} + \cdots + r_m(z)\bar{z}^m,$$

where each r_i is a rational function with poles off X. These modules arise in a natural fashion when one attempts to study rational approximation in Lip β norm. Our approach is based upon a novel use of the Cauchy transform. We define the transform $\hat{T}$ whenever T is a distribution with compact support; $\hat{T}$ is another distribution. The Key Lemma (Sect. 2) states that for certain kinds of spaces V of C^∞ functions, T annihilates $V + V\bar{z}$ if and only if $\hat{T}$ annihilates V. This fact, combined with certain estimates (Lemmas 4 and 6), leads to our main results, Theorem 1 (Sect. 3) and Theorem 2 (Sect. 4). Theorem 1 shows how uniform approximation theorems yield Lip α approximation theorems ($0 < \alpha < 1$). Theorem 2 shows that for many sets the general problem of Lip β approximation for nonintegral β can be reduced to the case $0 < \beta < 1$. In formulating Theorem 2 we set up the spaces $J_m(X, a)$ of bounded point derivations on the algebras $D^m(X)$, and this leads to Theorem 3 (Sect. 5), which gives a condition for failure of approximation in integral Lipshitz norms. The discussion of Section 6 is concerned with a useful integral representation for the Cauchy transform of an element of $(\text{Lip}\,\alpha)^*$ ($0 < \alpha < 1$).

* Supported by NSF Grant No. GP-33693X.

The techniques developed here have wide application, to approximation in other norms and to partial differential equations.

2. Preliminaries

We identify $\mathbb{C}$ with $\mathbb{R}^2$, and denote by $\mathscr{E}$ and $\mathscr{D}$ the usual linear topological spaces of complex-valued C^∞ functions on $\mathbb{C}$. Their duals $\mathscr{D}'$ and $\mathscr{E}'$ are, respectively, the space of distributions and the space of distributions with compact support [13]. The Cauchy transform $\hat{\varphi}$ of a function $\varphi \in \mathscr{D}$ is defined by

$$\hat{\varphi}(z) = \frac{1}{\pi} \int \frac{\varphi(w)}{z - w} \, d\mathscr{L}^2 w$$

for all $z \in \mathbb{C}$, where $\mathscr{L}^2$ is Lebesgue measure on $\mathbb{C}$. The linear map $\varphi \to \hat{\varphi}$ maps $\mathscr{D}$ continuously into $\mathscr{E}$. This allows us to define the Cauchy transform of an element of $\mathscr{E}'$. For $T \in \mathscr{E}'$ and $\varphi \in \mathscr{D}$ we set

$$\hat{T}(\varphi) = -T(\hat{\varphi}).$$

Then $\hat{T} \in \mathscr{D}'$ (in fact it may be seen that $\hat{T}$ is a temperate distribution [13]). We use the symbol $\bar{\partial}$ for the operator

$$(\partial/\partial x) + i(\partial/\partial y),$$

which may be applied to functions or distributions. We summarize the basic properties of ^ and $\bar{\partial}$ in a lemma, in which the various assertions are either classical or easy.

Lemma 1. $\bar{\partial}\hat{\varphi} = \varphi = \widehat{\bar{\partial}\varphi}$ *for* $\varphi \in \mathscr{D}$.

(ii) $\bar{\partial}\hat{S} = S = \widehat{\bar{\partial}S}$ *for* $S \in \mathscr{E}'$.

(iii) *The map* ^ : $\mathscr{E}' \to \mathscr{D}'$ *is a continuous linear injection with dense image.*

We let $\mathscr{P}_m$ denote the space of analytic polynomials of degree m or less, and

$$\mathscr{P} = \bigcup_{m=0}^{\infty} \mathscr{P}_m .$$

Given a compact set $X \subset \mathbb{C}$, $\tilde{\mathscr{R}}(X)$ is the space of all functions $f \in \mathscr{E}$ which are analytic on some neighborhood of X, and $\mathscr{R}(X)$ is the space of all functions $f \in \mathscr{E}$ which coincide on some neighborhood of X

with a rational function (in either case the neighborhood may depend on the function f). Observe that if $T \in \mathscr{E}' \cap \mathscr{R}(X)^{\perp}$, then spt $T \subset X$. The most general form of *Runge's theorem* states: *If* $T \in \mathscr{E}'$, *then* $T \perp \tilde{\mathscr{R}}(X)$ *if and only if* $T \perp \mathscr{R}(X)$. It is readily seen that *a given distribution* $T \in \mathscr{E}'$ *annihilates* $\tilde{\mathscr{R}}(X)$ *if and only if* spt $\hat{T} \subset X$. Hence $T \perp \mathscr{R}(X)$ *if and only if* spt $\hat{T} \subset X$.

LEMMA 2. *Let V be a linear subspace of $\mathscr{E}$ such that for each $v \in V$ the following three conditions hold.*

(a) $\bar{\partial} v \in V$,

(b) $\bar{z}\, \bar{\partial} v \in V$,

(c) *There exists* $n \in \mathbb{Z}^+ = \mathbb{Z} \cap \{n \geqslant 0\}$ *(depending on v) such that* $(\bar{\partial})^n v = 0$.

Then for every $T \in \mathscr{E}'$ the following are equivalent.

(1) $T \perp V$.

(2) $\bar{\partial} T \perp V + V\bar{z}$.

Proof. Suppose (1) holds, and let $u + \bar{z}v \in V + \bar{z}V$. Then $(\bar{\partial} T)(u + \bar{z}v) = -T(\bar{\partial} u + v + \bar{z}\,\bar{\partial} v) = 0$ by (a) and (b), hence (2) holds.

Conversely, suppose (2) holds, and let $v \in V$. We claim that for any $m \in \mathbb{Z}^+$, $(\bar{z})^m (\bar{\partial})^m v \in V$ and

$$Tv = [(-1)^m/m!]\; T[(\bar{z})^m (\bar{\partial})^m v].$$

The claim is established by induction on m. Clearly it is true for $m = 0$. Suppose it holds for a given $m \geqslant 0$. Then by (b), $\bar{z}\bar{\partial}(\bar{z}^m\, \bar{\partial}^m v) = m\bar{z}^m\, \bar{\partial}^m v + \bar{z}^{m+1}\, \bar{\partial}^{m+1} v \in V$, hence $\bar{z}^{m+1}\, \bar{\partial}^{m+1} v \in V$, and

$$\begin{aligned} 0 &= T[\bar{\partial}(\bar{z}^{m+1}\, \bar{\partial}^m v)] \\ &= T[(m+1)\, \bar{z}^m\, \bar{\partial}^m v + \bar{z}^{m+1}\, \bar{\partial}^{m+1} v] \\ &= (-1)^m (m+1)!\; T(v) + T[\bar{z}^{m+1}\, \bar{\partial}^{m+1} v], \end{aligned}$$

so the claim holds for $m + 1$ also.

Taking $m = n$ (cf. (c)), we conclude that $Tv = 0$. Thus $T \perp V$, and (1) holds.

KEY LEMMA. *Let V be a subspace of $\mathscr{E}$ which satisfies the conditions* (a), (b), (c) *of Lemma* 2. *Let $T \in \mathscr{E}'$, $X \subset \mathbb{C}$ be compact, and $\mathscr{R}(X) \subset V$. Then*

(1) $T \perp V + V\bar{z}$ *if and only if*

(2) $\hat{T} \perp V$.

Proof. Suppose (1) holds. Since $\mathscr{R}(X) \subset V$, it follows that spt $\hat{T} \subset X$, and in particular $\hat{T} \in \mathscr{E}'$, so that Lemma 2 applies, with T replaced by $\hat{T}$. Since $\bar{\partial}\hat{T} = T$, (2) holds.

Conversely, suppose (2) holds. Then $\hat{T} \perp \mathscr{R}(X)$, so spt $\hat{T} \subset X$, and thus $\hat{T} \in \mathscr{E}'$. Applying Lemma 2 again, we see that (1) holds.

3. Lip α, $0 < \alpha < 1$

If f is a complex-valued function defined on a subset E of $\mathbb{C}$, $r \in \mathbb{R}$, and $0 < \alpha \leqslant 1$, we set

$$\begin{aligned}
\omega(f, E, r) &= \sup\{|f(z) - f(w)| : z, w \in E, |z - w| \leqslant r\},\\
\|f\|_{\alpha,E} &= \sup\{r^{-\alpha}\omega(f, E, r) : r > 0\},\\
\mathrm{Lip}(\alpha, E) &= \{f \in \mathbb{C}^E : \|f\|_{\alpha,E} < \infty\},\\
\mathrm{lip}(\alpha, E) &= \{f \in \mathrm{Lip}(\alpha, E) : r^{-\alpha}\omega(f, E, r) \to 0 \text{ as } r \downarrow 0\}.
\end{aligned}$$

When endowed with the norm

$$\|f\|'_{\alpha,E} = \|f\|_{\alpha,E} + \|f\|_{u,E},$$

Lip(α, E) becomes a Banach algebra. Here $\|f\|_{u,E}$ is the uniform norm. The object of this section is to apply the key lemma to approximation in Lip(α, X) for $0 < \alpha < 1$ and compact X.

If $V \subset \mathrm{Lip}(\alpha, X)$, then $[V]_{\alpha,X}$ (or just $[V]_\alpha$) denotes the closure of V with respect to the norm $\|\cdot\|'_\alpha$, and $[V]_{u,X}$ denotes the uniform closure.

If T is an element of Lip$(\alpha, X)^*$, the continuous dual of Lip(α, X), then the restriction $T \mid \mathscr{E}$ is a distribution of order 1 with support in X. Hence we can form $(T \mid \mathscr{E})\hat{} \in \mathscr{D}'$, and we abbreviate this to $\hat{T}$. If $\hat{T} = 0$, then by Lemma 1 (iii), T annihilates $\mathscr{E}$, and hence T annihilates lip(α, X), since $\mathscr{E}$ is dense in lip(α, X) (for $0 < \alpha < 1$). Also, Runge's theorem implies that T annihilates $\mathscr{R}(X)$ if and only if T annihilates $\tilde{\mathscr{R}}(X)$, hence by the separation theorem, $[\mathscr{R}]_\alpha = [\tilde{\mathscr{R}}]_\alpha$.

The following result is essentially classical [2, 6, 10, 18].

LEMMA 4. *Let* $0 < \alpha < 1$. *Then there is a constant* K, *which depends only on* α, *such that*

$$\|\hat{\varphi}\|_{\alpha,\mathbb{C}} \leqslant K \|\varphi\|_u d^{1-\alpha}$$

whenever $\varphi \in \mathscr{D}$ *and* $d =$ diam spt φ.

Combining Lemma 4 and the F. Riesz representation theorem we obtain a representation of $\hat{T}$ for $T \in \mathrm{Lip}(\alpha, X)^*$. A more refined version is obtained in Section 6.

LEMMA 5. *Let* $0 < \alpha < 1$, *let* $X \subset \mathbb{C}$ *be compact, and let* $T \in \mathrm{Lip}(\alpha, X)^*$. *Then there is a complex Borel regular measure* μ *on* $\mathbb{C}$ *such that* $|\mu|(Y) < \infty$ *for all compact* Y *and*

$$\hat{T}\varphi = \int \varphi \, d\mu$$

whenever $\varphi \in \mathscr{D}$.

Now we can state and prove the main result of this section.

THEOREM 1. *Let* $0 < \alpha < 1$, *let* $m \in \mathbb{Z}^+$, *and let* $X \subset \mathbb{C}$ *be compact. If*

$$[\mathscr{R}\bar{\mathscr{P}}_m]_u = C(X),$$

then

$$[\mathscr{R}\bar{\mathscr{P}}_{m+1}]_\alpha = \mathrm{lip}(\alpha, X).$$

Proof. Suppose

$$[\mathscr{R}\bar{\mathscr{P}}_m]_u = C(X).$$

Let $T \in \mathrm{Lip}(\alpha, X)^*$, $T \perp \mathscr{R}\bar{\mathscr{P}}_{m+1}$. Then by the Key Lemma, $\hat{T} \perp \mathscr{R}\bar{\mathscr{P}}_m$, and by Lemma 5, $\hat{T}$ is represented on $\mathscr{D}$ by a finite Borel regular measure supported on X. Hence $\hat{T} = 0$, and so $T \perp \mathrm{lip}(\alpha, X)$. It follows that $\mathscr{R}\bar{\mathscr{P}}_{m+1}$ is dense in $\mathrm{lip}(\alpha, X)$.

EXAMPLE 1. In case $m = 0$ the theorem states that

$$[\mathscr{R}]_u = C(X) \tag{*1}$$

implies

$$[\mathscr{R} + \mathscr{R}\bar{z}]_\alpha = \mathrm{lip}(\alpha, X) \qquad (0 < \alpha < 1). \tag{*2}$$

The $\mathscr{R}\bar{z}$ cannot be removed, in general. In [15] a measure theoretic condition is given which is necessary and sufficient for

$$[\mathscr{R}]_\alpha = \mathrm{lip}(\alpha, X) \tag{*3}$$

to hold, and by using this condition an example is constructed in which (*1) holds and (*3) fails.

EXAMPLE 2. Vitushkin [19, 8, 10] has given a necessary and sufficient condition for (*1) to hold, in terms of analytic capacities. Using this, one can often check the validity of the hypothesis in case $m = 0$. In case $m > 0$ the problem of determining for which X one has

$$[\mathscr{R}\bar{\mathscr{P}}_m]_u = C(X)$$

has not been studied at all, as far as I know. Here we give an example of an X such that

$$[\mathscr{R} + \mathscr{R}\bar{z}]_u = C(X), \tag{*4}$$

whereas (*1) fails.

By combining [8, chap. VIII, Sect. 5.1; and 1 or 12] we see that there exist compact sets $X \subset \mathbb{C}$ such that (*1) fails and yet $\mathscr{R}(X)$ is dense in $L^3(X, \mathscr{L}^2)$ in $L^3(X)$ norm (here $L^3(X)$ is the usual space of $\mathscr{L}^2$ measurable functions f on X such that $\int |f|^3 \, d\mathscr{L}^2 < \infty$). Let X be such a set. We will show that (*4) holds.

Suppose μ is a finite Borel measure on X and $\mu \perp \mathscr{R} + \mathscr{R}\bar{z}$. Then for $1 \leqslant q < 2$ we have

$$\left[\int_X |\hat{\mu}|^q \, d\mathscr{L}^2\right]^{1/q} \leqslant \left[\int_X \left\{\int_X \frac{d|\mu|(w)}{|w - z|}\right\}^q d\mathscr{L}^2(z)\right]^{1/q}$$
$$\leqslant \int_X \left\{\int_X \frac{d\mathscr{L}^2(z)}{|w - z|^q}\right\}^{1/q} d|\mu|(w) \leqslant M \|\mu\|,$$

where M depends on diam X and q. Thus $\hat{\mu} \in L^{3/2}(X, \mathscr{L}^2) \cap \mathscr{R}^{\perp}$, and since $\mathscr{R}$ is dense in $L^3(X)$ and $L^3(X)^* = L^{3/2}(X)$ we infer that $\hat{\mu} = 0$, hence $\mu = 0$. Thus (*4) holds.

It is worth noting that the annular Swiss Cheese of Roth [16] has the property that

$$[\mathscr{R}]_u \neq [\mathscr{R} + \mathscr{R}\bar{z}]_u,$$

since

$$(7/64)(|z|^2 - 1) \notin [\mathscr{R}]_u.$$

However, it is not clear whether or not (*4) holds for this X.

EXAMPLE 3. It is easy to see that if int $X \neq \varnothing$, then

$$[\mathscr{R}]_u \neq [\mathscr{R}\bar{\mathscr{P}}_1]_u \neq [\mathscr{R}\bar{\mathscr{P}}_2] \neq \cdots.$$

EXAMPLE 4. If $\mathbb{C} \backslash X$ is connected, then

$$[\mathscr{R}]_u = [\mathscr{P}]_u, \qquad [\mathscr{R}]_\alpha = [\mathscr{P}]_\alpha,$$

and Mergelyan's theorem [8] tells us that

$$[\mathscr{P}]_u = C(X)$$

if and only if int $X = \varnothing$. Thus *if* $\mathbb{C}\backslash X$ *is connected and* int $X = \varnothing$, *then*

$$[\mathscr{P} + \mathscr{P}\bar{z}]_\alpha = \mathrm{lip}(\alpha, X) \qquad (0 < \alpha < 1).$$

4. Lip β, $\beta > 1$

The space $\mathrm{Lip}(\beta, \mathbb{C})$ (where $\beta = n + \alpha$, $1 \leqslant n \in \mathbb{Z}$ and $0 < \alpha \leqslant 1$) consists of all those bounded continuous functions on $\mathbb{C}$ which have bounded continuous partial derivatives of all kinds up to and including order n, and whose nth partial derivatives all belong to $\mathrm{Lip}(\alpha, \mathbb{C})$. The norm on $\mathrm{Lip}(\beta, \mathbb{C})$ is

$$\| f \|_\beta = \sum_{i+j \leqslant n} \left\| \frac{\partial^{i+j} f}{\partial x^i \, \partial y^j} \right\|_{u,\mathbb{C}} + \sum_{i+j=n} \left\| \frac{\partial^n f}{\partial x^i \, \partial y^j} \right\|_{\alpha,\mathbb{C}}.$$

If X is compact, then

$$I(X) = \{f \in \mathrm{Lip}(\beta, \mathbb{C}) : f \equiv 0 \text{ on } X\}$$

is a closed ideal in $\mathrm{Lip}(\beta, \mathbb{C})$, and we define

$$\mathrm{Lip}(\beta, X) = \mathrm{Lip}(\beta, \mathbb{C})/I(X),$$

with the quotient norm. We may think of $\mathrm{Lip}(\beta, X)$ as a space of functions on X: a function f on X corresponds to an element of $\mathrm{Lip}(\beta, X)$ if f has an extension in $\mathrm{Lip}(\alpha, \mathbb{C})$. (For a concrete description of $\mathrm{Lip}(\beta, X)$ in terms of local properties of f, see [18, Chap. VI].)

When we wish to distinguish, we will denote the coset $g + I(X) \in \mathrm{Lip}(\beta, X)$, corresponding to an element $g \in \mathrm{Lip}(\beta, \mathbb{C})$, by $\tilde{g}$.

The space $\mathrm{lip}(\beta, \mathbb{C})$ consists of those functions $f \in \mathrm{Lip}(\beta, \mathbb{C})$ whose nth partial derivatives belong to $\mathrm{lip}(\alpha, \mathbb{C})$, and $\mathrm{lip}(\beta, X)$ is the subspace of $\mathrm{Lip}(\beta, X)$ defined by

$$\mathrm{lip}(\beta, X) = [\mathrm{lip}(\beta, \mathbb{C}) + I(X)]/I(X).$$

Thus a function f defined on X corresponds to an element of $\mathrm{lip}(\beta, X)$ if f has an extension in $\mathrm{lip}(\beta, \mathbb{C})$.

We denote the quotient norm on $\mathrm{Lip}(\beta, X)$ by $\|f\|'_{\beta, X}$. Clearly $\|f\|'_{\beta,X}$ is dominated by the $C^{n+1}(K)$ norm of f whenever $f \in \mathscr{D}$ and K is an open disc containing X. The $C^{n+1}(K)$ norm of f is the sum

$$\sum_{i+j \leqslant n+1} \left\| \frac{\partial^{i+j} f}{\partial x^i \, \partial y^j} \right\|_{u,K}.$$

Using this fact and a smoothing argument we deduce that

$$[\mathscr{E}]_{\beta,X} = \mathrm{lip}(\beta, X)$$

for nonintegral β. In case $\beta = n + 1 \in \mathbb{Z}$ we denote

$$D^{n+1}(X) = [\mathscr{E}]_{n+1,X}.$$

Then $D^{n+1}(X)$ is a subalgebra of $\mathrm{Lip}(n + 1, X)$. Recall that if A is a complex algebra with unit, J is a maximal ideal of A, and $0 < p \in \mathbb{Z}$, then a pth *order derivation on A at J* is a linear functional $P\colon A \to \mathbb{C}$ which annihilates

$$J^p + \mathbb{C}.$$

For $1 \leqslant m \in \mathbb{Z}$, the maximal ideals of the algebra $D^m(X)$ are the sets

$$K(a) = \{f \in D^m(X) \colon f(a) = 0\},$$

corresponding to the various points $a \in X$. A derivation on $D^m(X)$ at $K(a)$ is called a *point derivation* at a. For $1 \leqslant m \in \mathbb{Z}$ we define

$$J_m(X, a)$$

as the vectorspace of bounded mth order point derivations on $D^m(X)$ at a.

At an isolated point of X, $J_m(X, a) = \{0\}$. At an accumulation point, the dimension of $J_m(X, a)$ lies between m and $\frac{1}{2}m(m + 3)$, and either value may be attained. We say X is *m-thick* if

$$\dim J_m(X, a) = \tfrac{1}{2}m(m + 3)$$

whenever $a \in X$. If this is the case, then all the partial derivatives

$$f \to \frac{\partial^{i+j} f}{\partial x^i \, \partial y^j}(a), \qquad (f \in \mathscr{E})$$

corresponding to $i + j \leqslant m$, extend to continuous linear functionals on $D^m(X)$. We denote the extensions by the symbols $D_{ij} \cdot (a)$. We say X is *uniformly m-thick* if each of the maps $a \to D_{ij} \cdot (a)$ is bounded on X, i.e. if there exists a constant $M > 0$ such that

$$| D_{ij} f(a)| \leqslant M \| f \|'_{m,X}$$

whenever $i + j \leqslant m$, $a \in X$, and $f \in D^m(X)$. For convenience we say that every compact set X is uniformly 0-thick. It is not hard to see that if X is uniformly m-thick, then $D_{ij} f(a)$ varies continuously with a for fixed $i, j \in \mathbb{Z}^+$ with $i + j \leqslant m$ and fixed $f \in D^m(X)$.

If every nonempty relatively open subset of X has positive area, then X is uniformly m-thick for every $m \in \mathbb{Z}^+$. The product $C \times C$ of any linear Cantor set with itself is uniformly m thick for every m. Thus there are uniformly m-thick sets with Hausdorff dimension zero.

It is possible to push through the ensuing results for certain sets X which are not m-thick, notably for C^β curves, but the simplest blanket assumption is m-thickness.

LEMMA 6. *Let $0 < \beta \notin \mathbb{Z}$ and $d > 0$. Then there is a constant $K > 0$, depending only on β and d, such that*

$$\| \hat{\varphi} \|_{\beta+1,\mathbb{C}} \leqslant K \| \varphi \|_{\beta,\mathbb{C}}$$

whenever $\varphi \in \mathscr{D}$ and diam spt $\varphi \leqslant d$.

This fact is widely known. It was shown to the author by C. Earle. It appears in [2, pp. 9–15] in case $0 < \beta < 1$.

THEOREM 2. *Let $X \subset \mathbb{C}$ be compact, $0 \leqslant m, n \in \mathbb{Z}, m < \beta < m + 1$. Consider the two conditions*:

1. $[\mathscr{R}\mathscr{P}_n]_\beta = \operatorname{lip}(\beta, X)$;
2. $[\mathscr{R}\mathscr{P}_{n+1}]_{\beta+1} = \operatorname{lip}(\beta + 1, X)$.

If X is uniformly m-thick, then (1) *implies* (2). *If X is uniformly* $(m + 1)$-*thick, then* (2) *implies* (1).

Proof. Suppose X is uniformly m-thick, and (1) holds. Let $T \in \operatorname{Lip}(\beta + 1, X)^*$ be an annihilator of $\mathscr{R}\mathscr{P}_{n+1}$. Then $\hat{T}$ is supported on X, $\hat{T} \perp \mathscr{R}\mathscr{P}_n$, and

$$| \hat{T}(\varphi)| \leqslant \| T \|_{\beta+1} K \| \varphi \|_{\beta,\mathbb{C}}$$

whenever $\varphi \in \mathscr{D}$, by Lemma 6. Here K depends only on β and diam X, which are fixed in the present discussion, so $\hat{T}$ is continuous with respect to the $\text{Lip}(\beta, \mathbb{C})$ norm.

Since X is m-thick, the Whitney–Calderón–Zygmund extension theorem [18, Chap. VI] implies that there exists a continuous linear map S: $\text{Lip}(\beta, X) \to \text{Lip}(\beta, \mathbb{C})$ such that

(a) $S\tilde{f} = f$ on X whenever $f \in \text{Lip}(\beta, \mathbb{C})$, and

(b) $S\tilde{f} \in \text{lip}(\beta, \mathbb{C})$ whenever $\tilde{f} \in \text{lip}(\beta, X)$. Thus for $f \in \mathscr{D}$ we have

$$|(\hat{T} \circ S)f| \leqslant \| \hat{T} \|_\beta \| S \|_\beta \| \tilde{f} \|'_{\beta, X},$$

so that $(\hat{T} \cdot S) \mid \mathscr{D}$ extends to a continuous linear functional on $\text{Lip}(\beta, X)$ (nonuniquely; the extension is only determined on $\text{lip}(\beta, X)$).

Fix $g \in \mathscr{R}\bar{\mathscr{P}}_n$. We wish to show that $(\hat{T} \circ S)(g) = 0$. Fix $\epsilon > 0$, and consider the function $h = S\tilde{g} - g \in \text{lip}(\beta, \mathbb{C})$. The various derivatives $D_{ij}h(a)$, corresponding to $i + j \leqslant m$, vary continuously on $\mathbb{C}$, and the top order derivatives are such that

$$[D_{ij}h(a) - D_{ij}h(b)]/|a - b|^\alpha \tag{*}$$

is continuous on $\mathbb{C} \times \mathbb{C}$. Since h vanishes identically on X and X is m-thick, it follows that all these derivatives vanish on X, while the functions (*) vanish on $X \times X$. Thus there is a closed neighborhood N of X such that

$$\| S\tilde{g} - g \|_{\beta, N} < \epsilon.$$

We may assume N is also m-thick, and apply the Whitney–Calderón–Zygmund theorem to obtain a function $k \in \text{Lip}(\beta, \mathbb{C})$ such that

$$k = S\tilde{g} - g \text{ on } N$$

and

$$\| k \|_{\beta, N} < K_1 \epsilon,$$

where K_1 is a constant which depends only on β and diam N. Thus

$$|(\hat{T} \cdot S)\tilde{g}| = |\hat{T}(S\tilde{g} - g)| = |\hat{T}k| \leqslant \| \hat{T} \|_\beta K_1 \epsilon,$$

and since this is true for every $\epsilon > 0$,

$$(\hat{T} \circ S)(\tilde{g}) = 0.$$

Hence $\hat{T} \circ S$ is an annihilator of $\mathscr{R}\bar{\mathscr{P}}_n$ in $\mathrm{Lip}(\beta, X)^*$, and so $\hat{T} \circ S = 0$ on $\mathrm{lip}(\beta, X)$ by the separation theorem and assumption (1).

Next we claim that $\hat{T} = 0$ on $\mathscr{D}$. To see this, fix $\varphi \in \mathscr{D}$ and $\epsilon > 0$. The function $S\tilde{\varphi} - \varphi$ belongs to $\mathrm{lip}(\beta, \mathbb{C})$, and as above there is a function $h \in \mathrm{lip}(\beta, \mathbb{C})$ such that $h = S\tilde{\varphi} - \varphi$ on a neighborhood of X, while $\| h \|_{\beta,\mathbb{C}} \leqslant K_1\epsilon$. Then

$$| \hat{T}(\varphi)| = | \hat{T}(S\tilde{\varphi} - \varphi)| = | \hat{T}h | \leqslant \| \hat{T} \| K_1\epsilon.$$

The claim follows.

Hence $\hat{T} \perp \mathscr{E}$, so $T \perp \mathscr{E}$, and $T \perp \mathrm{lip}(\beta, X)$. So (2) follows by the separation theorem.

The second assertion is proved in a similar way, except that the trivial estimate

$$\| \bar{\partial}\varphi \|_{\beta,\mathbb{C}} \leqslant 2 \| \varphi \|_{\beta+1,\mathbb{C}}$$

is used instead of Lemma 6. We omit the details.

COROLLARY. *Let $X \subset \mathbb{C}$ be compact, $0 < \alpha < 1$, $n \in \mathbb{Z}^+$. Suppose*

$$[\mathscr{R}\bar{\mathscr{P}}_n]_\alpha = \mathrm{lip}(\alpha, X).$$

Then

$$[\mathscr{R}\bar{\mathscr{P}}_{n+1}]_{1+\alpha} = \mathrm{lip}(1 + \alpha, X),$$

and, a fortiori,

$$[\mathscr{R}\bar{\mathscr{P}}_{n+1}]_1 = D^1(X).$$

EXAMPLE 5. If X has zero area, then

$$[\mathscr{R}]_\alpha = \mathrm{lip}(\alpha, X)$$

(cf. [15] or Sect. 6), hence

$$[\mathscr{R} + \mathscr{R}\bar{z}]_1 = D^1(X).$$

On the other hand there are many sets X with zero area for which

$$[\mathscr{R}]_1 \neq D^1(X).$$

In fact $\mathscr{R}$ is dense in $D^1(X)$ if and only if X is a subset of a finite disjoint union of simple C^1 curves [14].

EXAMPLE 6. If $[\mathscr{R}]_u = C(X)$, then

$$[\mathscr{R} + \mathscr{R}\bar{z} + \mathscr{R}\bar{z}^2]_1 = D^1(X).$$

I do not know an example for which $[\mathscr{R}]_u = C(X)$ and $[\mathscr{R} + \mathscr{R}\bar{z}]_1 \neq D^1(X)$.

EXAMPLE 7. Let X be such that

$$[\mathscr{R}]_u \neq [\mathscr{R} + \mathscr{R}\bar{z}]_u = C(X)$$

(cf. Example 2). Then

$$[\mathscr{R}\bar{\mathscr{P}}_3]_1 = D^1(X).$$

There are sets X of this type which are 1-thick, and for these X one can show that

$$[\mathscr{R} + \mathscr{R}\bar{z}]_1 \neq D^1(X).$$

(For more on this example, cf. Sect. 6.)

5. $J_m(X, a)$

In this section we give a result concerning approximation in integral Lipshitz norms.

THEOREM 3. *Let X be compact in $\mathbb{C}$, let $m, j \in \mathbb{Z}^+$, $j < m$, and suppose there exists a point $a \in X$ such that*

$$\dim J_m(X, a) > (j + 1)m - \tfrac{1}{2}j(j - 1).$$

Then

$$[\mathscr{R}\bar{\mathscr{P}}_j]_m \neq D^m(X).$$

Proof. For a function $f \in \mathscr{E}$, consider the polynomial

$$\pi(f) = \sum_{r=1}^{m} \sum_{s=0}^{r} \binom{r}{s} \frac{\partial^r f(a)}{\partial x^s\, \partial y^{r-s}} (x - a_1)^s (y - a_2)^{r-s},$$

where $a = a_1 + ia_2$. The linear function π maps $\mathscr{E}$ onto the space $\mathbb{P}_m$ of polynomials in $(x - a_1)$ and $(y - a_2)$ of degree m or less with no constant term. We may regard $\mathbb{P}_m$ as a subspace of $\mathscr{E}$, and then we may write $Tf = T\pi(f)$ whenever $f \in \mathscr{E}$ and T is a continuous mth order point derivation on $\mathscr{E}$ at a. Let

$$\mathscr{K} = \{f \in \mathscr{E} : |x - a|^{-m} f(z) \to 0 \text{ as } |z - a| \downarrow 0, z \in X\}.$$

Then $\mathscr{K}$ is a subspace of $\mathscr{E}$ and it is easy to see that

$$\mathscr{K} = \bigcap \{\mathscr{E} \cap \ker T\colon T \in J_m(X, a)\}.$$

This means that every $T \in J_m(X, a)$ factors through $\mathscr{E}/\mathscr{K}$. Let $K = \mathscr{K} \cap \mathbb{P}_m$. Then $J_m(X, a)$ is isomorphic to the dual of $\mathbb{P}_m/K$. Hence

$$\dim(\mathbb{P}_m/K) > (j+1)m - \tfrac{1}{2}j(j-1) = \tau, \text{ say.}$$

If $f \in \mathscr{R}\bar{\mathscr{P}}_j$, then $(\bar{\partial})^{j+1} f(a) = 0$, i.e.

$$\sum_{r=0}^{j+1} \binom{j+1}{r} i^{j+1-r} \frac{\partial^{j+1} f(a)}{\partial x^r\, \partial y^{j+1-r}} = 0.$$

It follows that the dimension of $\pi(\mathscr{R}\bar{\mathscr{P}}_j)$ is τ. Hence the dimension of

$$W = \frac{\pi(\mathscr{R}\bar{\mathscr{P}}_j) + K}{K}$$

does not exceed τ, so that W is a proper subspace of $\mathbb{P}_m/K$. If we now choose $T \in J_m(X, a)$ corresponding to a nonzero annihilator of W in $(\mathbb{P}_m/K)^*$, it follows that T is a nonzero annihilator of $\mathscr{R}\bar{\mathscr{P}}_j$ in $D_m(X)^*$. Hence $\mathscr{R}\bar{\mathscr{P}}_j$ is not dense.

Example 8. We observe that the hypotheses are fulfilled with $j = m - 1$ for any compact set X with $\mathscr{L}^2(X) > 0$, because the dimension of $J_m(X, a)$ is $\tfrac{1}{2}m(m+3)$ at every point a of full area density of X. Hence, if $\mathscr{L}^2(X) > 0$, then

$$[\mathscr{R}\bar{\mathscr{P}}_{m-1}]_m \neq D^m(X).$$

Example 9. It is possible that there exist *first order* bounded point derivations on $D^2(X)$ which do not extend continuously to $D^1(X)$. Let f be the function defined by

$$f(x) = \begin{cases} 0, & -1 \leqslant x \leqslant 0, \\ x^2, & 0 \leqslant x \leqslant 1, \end{cases}$$

and let $X = \{x + if(x) : -1 \leqslant x \leqslant 1\}$ be the graph of f. Then $J_1(X, 0)$ is the span of $\{D_1\}$, whereas $J_2(X, 0)$ is the span of

$$\{D_1, D_2, D_{20}\}.$$

Hence

$$[\mathscr{R}]_2 \neq D^2(X),$$

whereas $[\mathscr{R}]_1 = D^1(X)$, since X is a C^1 curve.

6. Representation of $\hat{T}$

In this section we show that for $T \in \mathrm{Lip}(\alpha, X)^*$, $0 < \alpha < 1$, the measure $\hat{T}$ is absolutely continuous with respect to area $\mathscr{L}^2$, and we give an explicit representation for $\hat{T}$. We show how this representation may be applied to give further results on approximation.

Fix $0 < \alpha < 1$, X compact in $\mathbb{C}$, and $T \in \mathrm{Lip}(\alpha, X)^* \cap \mathbb{C}^\perp$. If $f \in \mathrm{lip}(\alpha, X)$, then the function

$$(\rho f)(x, y) = \frac{f(x) - f(y)}{|x - y|^\alpha}$$

is continuous on $X \times X$, and ρ is an isometric injection of $\mathrm{lip}(\alpha, X)/\mathbb{C}$ into $C(X \times X)$. By the Hahn–Banach theorem and the Riesz representation theorem, there exists a finite complex Borel regular measure μ on $X \times X$ such that

$$Tf = \int \rho f \, d\mu$$

whenever $f \in \mathrm{lip}(\alpha, X)$. This construction goes back to De Leeuw [5]. Let $\varphi \in \mathscr{D}$. Then

$$\begin{aligned}\hat{T}(\varphi) = -T(\hat{\varphi}) &= -\int \frac{\hat{\varphi}(x) - \hat{\varphi}(y)}{|x - y|^\alpha} \, d\mu(x, y) \\ &= -\iint \frac{\varphi(\zeta)(x - y)}{(\zeta - x)(\zeta - y)\, |x - y|^\alpha} \, d\mathscr{L}^2(\zeta) \, d\mu(x, y) \\ &= -\int \varphi(\zeta) \left\{ \int \frac{(x - y)\, |x - y|^{-\alpha}}{(\zeta - x)(\zeta - y)} \, d\mu(x, y) \right\} d\mathscr{L}^2(\zeta).\end{aligned}$$

The use of Fubini's theorem is justified by the fact that we may put in absolute values in the second line, and get something bounded by $K \|\varphi\|_u$, where K depends only on α and diam spt φ (this estimate is essentially the same as Lemma 4). In fact, this step is permissible for $\varphi \in L^\infty(\mathscr{L}^2)$ with spt φ compact. Thus the expression in chain brackets is in $L^1_{\mathrm{loc}}(\mathscr{L}^2)$, regarded as a function of ζ. If we denote this expression by $\hat{T}(\zeta)$ (abusing the notation), we have

$$\hat{T}(\varphi) = -\int \varphi(\zeta) \, \hat{T}(\zeta) \, d\mathscr{L}^2\zeta.$$

Observe that if $\zeta \notin X$, $\psi \in \mathscr{E}$, and

$$\psi(x) = 1/(x - \zeta)$$

for all x near X, then $\hat{T}(\zeta) = \hat{T}(\psi)$. Hence if $T \perp \mathscr{R}(X)$, then $\hat{T}(\zeta) = 0$ for $\zeta \notin X$. This provides an elegant proof of the extended *Hartogs–Rosenthal theorem*: *If* $\mathscr{L}^2(X) = 0$, *then* $[\mathscr{R}]_\alpha = \mathrm{lip}(\alpha, X)$.

The first application is an extension of a theorem of Davie [4]. Davie's theorem asserts that for any compact set X, with boundary Y, we have

$$[A(X) + \mathscr{R}(Y)]_{u,Y} = C(Y),$$

where $A(X)$ denotes the collection of all continuous functions on X which are analytic on the interior of X. We strengthen this result in three ways: We replace $A(X)$ by a smaller space $B(Y)$, replace the uniform norm by the larger Lip α norm, and throw away X. If g is any bounded Borel function on $\mathbb{C}$ such that the set

$$\{x \in \mathbb{C} : g(x) \neq 0\}$$

is bounded, we define

$$\hat{g}(z) = \frac{1}{\pi} \int \frac{g(\zeta)}{z - \zeta}\, d\mathscr{L}^2\zeta.$$

From Lemma 4, and the fact that $\mathscr{D}$ is weak star dense in $L^\infty(\mathscr{L}^2)$, it is clear that $\hat{g} \in \mathrm{lip}(\alpha, \mathbb{C})$ for $0 < \alpha < 1$. For $T \in \mathrm{Lip}(\alpha, \mathbb{C})^* \cap \mathscr{E}' \cap \mathbb{C}^\perp$ it is easy to see that the formulas

$$T(\hat{g}) = -\hat{T}(g) = -\int \hat{T}(z)\, g(z)\, d\mathscr{L}^2 z$$

are valid. For any compact set $Y \subset \mathbb{C}$ we define the vectorspace $B(Y)$ by setting

$$B(Y) = \{\hat{g} : g \text{ is a bounded Borel function, } g = 0 \text{ off } Y\}.$$

THEOREM 4. *Let* $0 < \alpha < 1$, *and let* $Y \subset \mathbb{C}$ *be compact. Then*

$$[B(Y) + \mathscr{R}(Y)]_{\alpha,Y} = \mathrm{lip}(\alpha, Y).$$

Proof. Let $T \in \mathrm{Lip}(\alpha, Y)^*$, $T \perp B(Y) + \mathscr{R}(Y)$. Then $\hat{T}(z) = 0$ for $z \in \mathbb{C} \backslash Y$. Further, for every bounded Borel function g which vanishes off Y, we have

$$0 = T(\hat{g}) = -\int \hat{T}(z)\, g(z)\, d\mathscr{L}^2 z,$$

so that $\hat{T}(z) = 0$ for $\mathscr{L}^2$ almost all $z \in Y$. Hence $\hat{T} = 0$, so that $T \perp \mathrm{lip}(\alpha, Y)$.

The second application shows a relation between Lipshitz approximation and L^p approximation. Recall (Example 2) that $\hat{\mu} \in L^q_{\text{loc}}$ whenever μ is a measure with compact support and $1 \leqslant q < 2$. We will show that an analogous result holds for the transforms of elements of $\text{Lip}(\alpha, X)^*$.

Let $T \in \text{Lip}(\alpha, X)^*$ $(0 < \alpha < 1$, X compact), let $T \perp \mathbb{C}$, and let μ be a measure on $X \times X$ which represents T. Then for $q \geqslant 1$ we have

$$\begin{aligned} \| \hat{T} \|_{L^q(X)} &= \left[\int_X | \hat{T}(\zeta) |^q \, d\mathscr{L}^2\zeta \right]^{1/q} \\ &= \left[\int_X \left| \int \frac{(x-y) \, | x - y |^{-\alpha}}{(\zeta - x)(\zeta - y)} \, d\mu(x, y) \right|^q d\mathscr{L}^2\zeta \right]^{1/q} \\ &\leqslant \int | x - y |^{1-\alpha} \left\{ \int_X \frac{d\mathscr{L}^2\zeta}{| \zeta - x |^q \, | \zeta - y |^q} \right\}^{1/q} d \, | \mu | \, (x, y). \end{aligned}$$

In case

$$1 \leqslant q < 2/(1 + \alpha),$$

the expression in chain brackets is bounded by

$$K \, | x - y |^{\alpha - 1},$$

uniformly in (x, y), and thus $\hat{T} \in L^q(X)$. *Let*

$$1 < q < 2(1 + \alpha),$$

$$(1/p) + (1/q) = 1,$$

and suppose $\mathscr{R}\bar{\mathscr{P}}_m$ *is dense in* $L^p(X)$. *Then,* applying the Key Lemma, we see that $\mathscr{R}\bar{\mathscr{P}}_{m+1}$ *is dense in* $\text{lip}(\alpha, X)$.

As an example, if X is chosen that $[\mathscr{R}]_u \neq C(X)$ and $\mathscr{R}$ is dense in $L^3(X)$, then

$$[\mathscr{R}]_{1/4} \neq [\mathscr{R} + \mathscr{R}\bar{z}]_{1/4} = \text{lip}(\tfrac{1}{4}, \mathbb{C}).$$

Applying the corollary to Theorem 2 we obtain $[\mathscr{R}\bar{\mathscr{P}}_2]_1 = D^1(X)$. If X is chosen to be 1-thick, then $[\mathscr{R}\bar{\mathscr{P}}_1]_1 \neq D^1(X)$ since $[\mathscr{R}]_u \neq C(X)$. It follows that $[\mathscr{R}]_1 \neq [\mathscr{R}\bar{\mathscr{P}}_1]_1$, so finally we obtain

$$[\mathscr{R}]_1 \neq [\mathscr{R} + \mathscr{R}\bar{z}]_1 \neq [\mathscr{R} + \mathscr{R}\bar{z} + \mathscr{R}z^2]_1 = D^1(X).$$

Acknowledgments

I am grateful to C. Earle, T. Gamelin, J. Garnett, and B. Weinstock for valuable conversation and correspondence.

REFERENCES

1. T. BAGBY, Quasi topologies and rational approximation, *J. Functional Analysis* **10** (1972), 259–268.
2. L. BERS, "Theory of Pseudo-analytic Functions," N.Y.U. Institute for Mathematics and Mechanics, New York, 1953.
3. A. BROWDER, "Introduction to Function Algebras," Benjamin, 1969.
4. A. M. DAVIE, Bounded approximation and Dirichlet sets, *J. Functional Analysis* **6** (1970), 460–467.
5. K. DE LEEUW, Banach spaces of Lipshitz functions, *Studia Math.* **21** (1961), 55–66.
6. E. P. DOLŽENKO, On removal of singularities of analytic functions, *Amer. Math. Soc. Transl.* (2) **97** (1971), 33–41.
7. C. EARLE, Manuscript notes on Teichmüller theory.
8. T. GAMELIN, "Uniform Algebras," Prentice–Hall, Englewood Cliffs, N.J., 1969.
9. T. GAMELIN AND J. GARNETT, Pointwise bounded approximation and Dirichlet algebras, *J. Functional Analysis* 8 (1971), 360–404.
10. J. GARNETT, "Analytic Capacity and Measure," Springer Lecture Notes No. 297, 1973.
11. L. I. HEDBERG, The Stone–Weierstrass theorem in Lipschitz algebras, *Ark. Mat.* 8 (1969), 63–72.
12. L. I. HEDBERG, Approximation in the mean by analytic functions, *Trans. Amer. Math. Soc.* **163** (1972), 157–171.
13. L. HÖRMANDER, "Linear Partial Differential Operators," Springer, New York, 1969.
14. A. O'FARRELL, Lip 1 rational approximation, *J. Lond. Math. Soc.*, to appear.
15. A. O'FARRELL, Hausdorff content and rational approximation in fractional Lipschitz norms, to appear.
16. A. ROTH, Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen, *Comm. Math. Helv.* **11** (1938), 77–125.
17. D. R. SHERBERT, The structure of ideals and point derivations in Banach algebras of Lipshitz functions, *Trans. Amer. Math. Soc.* **111** (1964), 240–272.
18. E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions," Princeton, N.J. 1970.
19. A. G. VITUSHKIN, The analytic capacity of sets in problems of approximation theory, *Russian Math. Surveys* **22** (1967), no. 6, 139–200.